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2d, N=2 AND N=4 SUPERGRAVITY AND THE LIOUVILLE THEORY IN SUPERSPACE [1]

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Abstract

The two-dimensional (2d) manifestly locally supersymmetric actions describing the N=2 and N=4 extended super-Liouville theory coupled to the N=2 and N=4 conformal supergravity, respectively, are constructed in superspace. It is shown that the N=4 super-Liouville multiplet is described by the *improved* twisted-II scalar multiplet TM-II, whose kinetic terms are given by the $SU(2) \otimes U(1)$ WZNW model.

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1 *Introduction.* The non-critical N-extended superstrings are described by the 2d coupling of the N-extended supersymmetric scalar matter to the N-extended supergravity, with the exponential scalar potential. The standard bosonic (N=0) Liouville action is given by

$$I_0 = \frac{1}{2}\int d^2x\sqrt{g}\left[g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + QR^{(2)}\phi - \mu^2 e^\phi\right\} \ . \tag{1}$$

This action is invariant under the local Weyl transformations

$$g_{\alpha\beta} \to e^{2\sigma}g_{a\beta} \ , \qquad \phi \to \phi - 2\sigma \ , \tag{2}$$

provided that the classical background charge $Q = 2$, which is equivalent to demanding the metric $e^\phi g_{\alpha\beta}$ to be invariant.

It is rather straightforward to generalize the action (1) to the case of rigid or local (1,1) supersymmetry [1, 2], but it is much less obvious for the (2,2) and (4,4) *local* supersymmetry. The rigid manifestly supersymmetric actions describing the N=2 and N=4 super-Liouville theories are known [3, 4] but, to the best of my knowledge, their locally supersymmetric counterparts were not constructed yet. Accordingly, the known studies of the N=2 and N=4 non-critical strings [5, 6] were only performed in the superconformal gauge. It is the purpose of this letter to fill this gap. A manifestly covariant and supersymmetric formulation of the N=2 and N=4 supergravity, and of the related non-critical superstrings as well, can be useful for studies of super-Riemannian surfaces and super-Beltrami differentials, and for calculating the non-critical superstring amplitudes, where the superconformal gauge is not convenient and a light-cone gauge is not accessible.

2 *The N=2 Liouville Action.* The appropriate framework for describing (2,2) supersymmetric matter couplings to (2,2) supergravity is provided by (2,2) superspace. The necessary tools for that, such as superspace measures, invariant actions and component projection formulae, were recently developed by Grisaru and Wehlau [7].

The N=2 superspace has two real bosonic coordinates $x^{\mp\!\!\!\!\!\!\;+}$ and $x^{=}$, and two complex fermionic coordinates θ^{+} and θ^{-}, as well as their conjugates $\theta^{\dot{+}}$ and $\theta^{\dot{-}}$. In addition to the (2,2) superspace general coordinate transformations, the full local symmetries of the (nonminimal) (2,2) supergravity include the local Lorentz symmetry, an axial $U_{\rm A}(1)$ and a vector $U_{\rm V}(1)$ internal symmetries. The geometry of the (2,2) superfield supergravity is described in terms of the covariant spinorial derivatives

$$\nabla_{\pm} = E_{\pm}{}^{M}\partial_{M} + \Omega_{\pm}\mathcal{M} + \Gamma_{\pm}\mathcal{X} + \tilde{\Gamma}_{\pm}\tilde{\mathcal{X}} \ , \tag{3}$$

where the generators of the local Lorentz, $U_{\rm V}(1)$ and $U_{\rm A}(1)$ symmetries, $\mathcal{M}$, $\mathcal{X}$ and $\tilde{\mathcal{X}}$, respectively, have been introduced. The *minimal* $U_{\rm A}(1)$ version of the (2,2) supergravity is defined by the constraints (*cf.* ref. [8])

$$\begin{gathered}
\{\nabla_{\pm}\,,\nabla_{\pm}\} \ = \ 0 \ , \quad \{\nabla_{+}\,,\nabla_{\dot{+}}\} \ = \ i\nabla_{\mp\!\!\!\!\!\!\;+} \ , \quad \{\nabla_{-}\,,\nabla_{\dot{-}}\} \ = \ i\nabla_{=} \ , \quad \{\nabla_{+},\nabla_{\dot{-}}\} \ = \ 0 \ , \\
\{\nabla_{+},\nabla_{-}\} \ = \ -\tfrac{1}{2}R^{*}(\mathcal{M}-i\mathcal{X}) \ ,
\end{gathered} \tag{4}$$

which leave a 2d ‘graviton’, a complex ‘gravitino’ and a $U(1)$ ‘graviphoton’ as the only component gauge fields in the theory. The (2,2) supergravity constraints (4) allow the existence of the covariantly chiral superfields. It follows from the Bianchi identities that the superfield R is covariantly chiral, $\nabla_{\dot{\pm}}R=0$. The constraints (4) are also known to be invariant under the additional local Weyl (scale) transformations

$$E_{\pm} \to e^{\Sigma}E_{\pm} \ , \quad R^{*} \to e^{2\Sigma}(R^{*}+4[\nabla_{-},\nabla_{+}]\Sigma) \ , \tag{5}$$

where the Weyl (2,2) superfield parameter Σ can be expressed in terms of a (2,2) chiral superfield [7].

The ‘component’ complex curvature $R^{(2)}_{\rm c}$ of the (2,2) supergravity can be most easily determined from the ‘space-time’ commutator

$$\begin{aligned}
[\nabla_{\mp\!\!\!\!\!\!\;+},\nabla_{=}] = \ & \frac{1}{2}\left(\nabla^{2}R-\tfrac{1}{2}RR^{*}\right)(\mathcal{M}+i\mathcal{X}) - \frac{1}{2}\left(\overline{\nabla}^{2}R^{*}+\tfrac{1}{2}RR^{*}\right)(\mathcal{M}-i\mathcal{X}) \\
& +\frac{1}{2}(\nabla_{+}R)\nabla_{-} + \frac{1}{2}(\nabla_{-}R)\nabla_{+} - \frac{1}{2}(\nabla_{\dot{+}}R^{*})\nabla_{\dot{-}} - \frac{1}{2}(\nabla_{\dot{-}}R^{*})\nabla_{\dot{+}} \ ,
\end{aligned} \tag{6}$$

which is a consequence of the supergravity constraints (4). Therefore, we can identify

$$\left(\nabla^{2}R-\tfrac{1}{2}RR^{*}\right) = \mathcal{R} \ , \quad \mathcal{R}| = R^{(2)}_{\rm c} \ , \tag{7}$$

where | means the projection onto the leading component of the superfield (we ignore the gravitino contributions). The real part of the complex curvature $R^{(2)}_{\mathrm{c}}$ is just the usual 2d curvature $R^{(2)}$, whereas its imaginary part is the abelian field strength of the graviphoton gauge field from the (2,2) minimal supergravity multiplet.

We are now in a position to write down the manifestly supersymmetric superfield action generalizing that of eq. (1) to the case of (2,2) local supersymmetry. Since we are not interested in the most general matter couplings in (2,2) supergravity (see however, ref. [9]) but the Liouville-type interaction, we restrict ourselves to the (2,2) matter described by some chiral superfields ϕ^a. Their coupling to the (2,2) supergravity in (2,2) superspace is given by

$$I_2 = \int d^2x \left\{ \int d^2\theta d^2\bar{\theta} E^{-1} K(\phi,\bar{\phi}) - \int d^2\theta \mathcal{E}^{-1} W(\phi) - \int d^2\theta \mathcal{E}^{-1} R\Upsilon(\phi) + \text{h.c.} \right\} \ , \quad (8)$$

where the supervielbein determinant $E(x,\theta,\bar{\theta})$, the chiral density $\mathcal{E}(x,\theta)$, the Kähler potential $K(\phi,\bar{\phi})$, the superpotential $W(\phi)$, and the (holomorphic) dilaton $\Upsilon(\phi)$ have been introduced. In components, after using the projection formulae of ref. [7], the last term in eq. (8) yields

$$\int d^2x \int d^2\theta \mathcal{E}^{-1} R\Upsilon(\phi)\Big| = \int d^2x e^{-1} (\nabla^2 - \tfrac{1}{2}R^*) R\Upsilon\Big| = \quad (9)$$

$$= \int d^2x e^{-1} \left(\mathcal{R}\Upsilon + \tfrac{1}{2}RR^*\Upsilon + R\nabla^2\Upsilon - \tfrac{1}{2}RR^*\Upsilon \right)\Big| \equiv \int d^2x e^{-1} \left(\mathcal{R}\Upsilon + R\nabla^2\Upsilon \right)\Big| \ ,$$

which proves that the non-propagating complex auxiliary field $H = R|$ of the (2,2) supergravity multiplet enters the action (8) *linearly*. It is also clear from eq. (9) that the real part of the dilaton is coupled to the 2d curvature, whereas its imaginary part is coupled to the abelian field strength of the graviphoton. Evaluating the potential terms for the action (8) is easy, *viz.*

$$2\frac{\partial^2 K}{\partial\phi^a \partial\bar{\phi}^b} F^a F^{*b} + \frac{1}{2} H^* W + F^a \frac{\partial W}{\partial \phi^a} + H F^a \frac{\partial \Upsilon}{\partial \phi^a} + \text{h.c.} \ , \quad (10)$$

and it leads, after eliminating the matter complex auxiliary fields F, to the constraint

$$W = \left[\frac{\partial^2 K}{\partial\phi^a \partial\bar{\phi}^b} \right]^{-1} \frac{\partial \Upsilon^*}{\partial\bar{\phi}^b} \left(\frac{\partial W}{\partial\phi^a} + H \frac{\partial\Upsilon}{\partial\phi^a} \right) \ , \quad (11)$$

relating the superpotential with the dilaton, without using any gauge (*cf.* ref. [5]). In the case of a *single* chiral superfield ϕ, it is always possible to make the dilaton field linear, $\Upsilon = \phi$, by field redefinition. Then eq. (11) forces the Kähler metric to be flat, $K = \bar{\phi}\phi$, and implies $W(\phi) = \mu e^{\phi} + H$, like in ref. [5]. We can therefore conclude that the action (8) is the (2,2) locally supersymmetric generalization of eq. (1) indeed.

3 *The N=4 Liouville Action.* After the relatively simple N=2 exercise given above, I turn to a construction of the N=4 generalization of the Liouville action (1) in the curved superspace of 2d, (4,4) supergravity.

The $N = 4$ superspace is parametrized by the coordinates

$$z^{A} \;=\; (x^{+\!\!\!+}, x^{=}, \theta^{+i}, \theta^{-i}, \theta^{\overset{\bullet}{+}}{}_{i}, \theta^{\overset{\bullet}{-}}{}_{i}) \;, \tag{12}$$

where $x^{+\!\!\!+}$ and $x^{=}$ are two real bosonic coordinates, $\theta^{\pm i}$ and their complex conjugates $\theta^{\overset{\bullet}{\pm}}{}_{i}$ are complex fermionic coordinates, $i = 1, 2$. The fermionic coordinates $\theta^{\pm i}$ are spinors with respect to $SU(2)$. Their complex conjugates are defined by

$$(\theta^{\pm i})^{*} \;\equiv\; \theta^{\overset{\bullet}{\pm}}{}_{i} \quad , \quad \theta^{\overset{\bullet}{\pm}\, i} = \mathcal{C}^{ij}\theta^{\overset{\bullet}{\pm}}{}_{j} \quad , \tag{13}$$

where the star denotes usual complex conjugation. The $SU(2)$ indices are raised and lowered by $\mathcal{C}^{ij}$ and $\mathcal{C}_{ij}$, whose explicit form is given by $\mathcal{C}^{ij} = i\varepsilon^{ij}$ and $(\mathcal{C}^{ij})^{*} = \mathcal{C}_{ij}$.

The local symmetries of the minimal (4,4) superfield supergravity are the (4,4) superspace general coordinate transformations, local Lorentz frame rotations and $SU(2)$ internal frame rotations. Therefore, the fully covariant derivatives in the curved (4,4) superspace should include the tangent space generators for all that symmetries with the corresponding connections,

$$\nabla_{A} \;=\; E^{M}_{A} D_{M} + \Omega_{A}\mathcal{M} + i\;\Gamma_{A}\cdot\mathcal{Y} \;, \tag{14}$$

where the (4,4) supervielbein E^{M}_{A}, the Lorentz generator $\mathcal{M}$ with the Lorentz connection Ω_{A}, and the $SU(2)$ generators $\mathcal{Y}_{i}{}^{j}$ with the $SU(2)$ connection $(\Gamma_{A})_{j}{}^{i}$ have been introduced. We use the notation $\Gamma_{A}\cdot\mathcal{Y} \equiv (\Gamma_{A})_{j}{}^{i}\mathcal{Y}_{i}{}^{j}$.

Assuming that the supervielbein is invertible, the lowest-order component in the θ-expansion of the superfield E_μ^a can be identified with the zweibein, $E_\mu^a\Big| = e_\mu^a$. Similarly, $E_\mu^{i\pm}\Big| = \psi_\mu^{i\pm}$ and $\Gamma_{\mu j}{}^i\big| = B_{\mu j}{}^i$ define the rest of the gauge fields in the (4,4) supergravity multiplet. The superfield torsion and curvature tensors are defined by

$$[\nabla_A, \nabla_B\} = T_{AB}{}^C \nabla_C + R_{AB}\mathcal{M} + i\ F_{AB} \cdot \mathcal{Y}\ . \tag{15}$$

The generators for the local Lorentz and $SU(2)$ frame transformations can be defined, e.g., by their action on the spinorial derivatives,

$$\begin{aligned}
[\mathcal{M}, \nabla_{\pm i}] &= \pm\tfrac{1}{2}\nabla_{\pm i}\ , \quad [\mathcal{M}, \nabla_{\dot{\pm}}{}^i] = \pm\tfrac{1}{2}\nabla_{\dot{\pm}}{}^i\ , \\
[\mathcal{Y}_i{}^j, \nabla_{\pm k}] &= +\,\delta_k{}^j\nabla_{\pm i} - \tfrac{1}{2}\delta_i{}^j\nabla_{\pm k}\ , \\
[\mathcal{Y}_i{}^j, \nabla_{\dot{\pm}}{}^k] &= -\,\delta_i{}^k\nabla_{\dot{\pm}}{}^j + \tfrac{1}{2}\delta_i{}^j\nabla_{\dot{\pm}}{}^k\ .
\end{aligned} \tag{16}$$

The superspace constraints defining the (4,4) minimal supergravity were first formulated by Gates *et. al.* in ref. [10]. In our notation, they take the form

$$\begin{aligned}
\{\nabla_{\pm i}, \nabla_{\pm j}\} &= 0\ , \quad \{\nabla_{+i}, \nabla_{\dot{+}j}\} = i\mathcal{C}_{ij}\nabla_{+\!\!\!+}\ , \quad \{\nabla_{-i}, \nabla_{\dot{-}j}\} = i\mathcal{C}_{ij}\nabla_{=}\ , \\
\{\nabla_{+i}, \nabla_{-}{}^j\} &= -\,\tfrac{i}{2}\ R^*\ \left(\delta_i{}^j\mathcal{M} - \mathcal{Y}_i{}^j\right)\ , \\
\{\nabla_{+i}, \nabla_{\dot{-}}{}^j\} &= -\,\tfrac{i}{2}\ S\ \left(\delta_i{}^j\mathcal{M} - \mathcal{Y}_i{}^j\right) - \tfrac{1}{2}\ T\ \left(\delta_i{}^j\mathcal{M} - \mathcal{Y}_i{}^j\right)\ ,
\end{aligned} \tag{17}$$

plus their complex conjugates, where the four scalar (4,4) covariant field strength superfields have been introduced, the complex one, R, and the two real ones, S and T. Using the Bianchi identities associated with the constraints (17), one finds that those superfields satisfy the constraints [10, 11]

$$\nabla_{\dot{\pm}i}R = 0\ , \quad \nabla_{\pm i}R = \pm\ \nabla_{\dot{\pm}i}S\ , \quad \nabla_{\pm i}S = \pm\ i\ \nabla_{\pm i}T\ , \tag{18}$$

which are the defining constraints of the 2d, (4,4) *twisted*-I off-shell hypermultiplet (TM-I) according to the classification of ref. [12]. The independent components of the TM-II are determined by the constraints (18) and they comprise, in addition to the leading scalars (R, R^*, S, T), a complex spinor $SU(2)$ doublet $\psi_\pm^i$, a real auxiliary

singlet A and an auxiliary triplet $A_i{}^j$. The supersymmetry transformation laws of the TM-I components can be found, e.g., in ref. [12]. This 2d, (4,4) off-shell hypermultiplet can be obtained via dimensional reduction from the 4d, N=2 off-shell vector multiplet [13].

As far as the 'component' quaternionic curvature of the (4,4) supergravity is concerned, it can be deduced from the 'space-time' commutator [3]

$$
\begin{aligned}
[\nabla_{\ddagger}, \nabla_{=}] \; = \; & \frac{i}{4}\Big[-(\nabla_{-}{}^{i}R)\nabla_{+i} + (\nabla_{+}{}^{i}R)\nabla_{-i} + (\nabla_{\underset{-}{\bullet}}{}^{i}R^{*})\nabla_{\underset{+i}{\bullet}} - (\nabla_{\underset{+}{\bullet}}{}^{i}R^{*})\nabla_{\underset{-i}{\bullet}} \\
& \qquad + (\nabla_{\underset{-}{\bullet}}{}^{i}\,(S+iT))\nabla_{+i} - (\nabla_{-}{}^{i}\,(S-iT))\nabla_{\underset{+i}{\bullet}} \\
& \qquad + (\nabla_{\underset{+}{\bullet}}{}^{i}\,(S-iT))\nabla_{-i} - (\nabla_{+}{}^{i}\,(S+iT))\nabla_{\underset{-i}{\bullet}}\Big] \\
& + \frac{1}{2}\Big[RR^{*} - S^{2} - T^{2} + \tfrac{i}{4}(\nabla_{+}{}^{i}\nabla_{-i}R) - \tfrac{i}{4}(\nabla_{\underset{+}{\bullet}}{}^{i}\nabla_{\underset{-i}{\bullet}}R^{*}) \\
& \qquad + \tfrac{i}{4}(\nabla_{\underset{+}{\bullet}}{}^{i}\nabla_{-i}\,(S-iT)) - \tfrac{i}{4}(\nabla_{+}{}^{i}\nabla_{\underset{-i}{\bullet}}\,(S+iT))\Big]\mathcal{M} \\
& + \frac{i}{8}\Big[(\nabla_{+}{}^{i}\nabla_{-j}R) - (\nabla_{\underset{+}{\bullet}}{}^{i}\nabla_{\underset{-j}{\bullet}}R^{*}) \\
& \qquad + (\nabla_{\underset{+}{\bullet}}{}^{i}\nabla_{-j}\,(S-iT)) - (\nabla_{+}{}^{i}\nabla_{\underset{-j}{\bullet}}\,(S+iT))\Big]\mathcal{Y}_{i}{}^{j}\ , \\
& [\nabla_{\ddagger}, \nabla_{\ddagger}] \; = \; 0\ , \qquad [\nabla_{=}, \nabla_{=}] \; = \; 0\ .
\end{aligned} \tag{19}
$$

The real part of the quaternionic curvature (in front of $\mathcal{M}$) contains $R^{(2)}$ in its leading component, whereas the imaginary part (in front of the $SU(2)$ generator $\mathcal{Y}_i{}^j$) has the $SU(2)$ field strength $F_j{}^i(B)$ in its leading component.

There exist two known $(8+8)$ off-shell versions of 2d, (4,4) hypermultiplet (with finite number of auxiliary fields), TM-I and TM-II. The TM-I was already introduced above. The *twisted*-II off-shell hypermultiplet (TM-II) in 2d was discovered by Ivanov and Krivonos [4]. The covariant superspace constraints defining this version of (4,4) hypermultiplet are given by

$$
\nabla_{\pm i}\, L_j{}^k = \mp i\left(\delta_i{}^k\nabla_{\pm j}P - \tfrac{1}{2}\delta_j{}^k\nabla_{\pm i}P\right)\ , \tag{20}
$$

in terms of 1+3 scalar superfields P and $L_i{}^j$, where $P^* = P$, $L_i{}^j = (L_j{}^i)^*$ and $L_i{}^i = 0$. It follows from eq. (20) that the independent components of TM-II comprise, in

[3]The full superspace structure of 2d, (4,4) supergravity will be considered in a separate publication [11].

addition to the leading components P and $L_i{}^j$, a complex spinor doublet $\lambda^i_{\pm}$, and four auxiliary scalars: a complex one G and two real ones M and N. The supersymmetry transformation laws of the TM-II components can be found e.g., in ref. [12]. The TM-II can be obtained via dimensional reduction from the 4d, N=2 off-shell tensor multiplet.

Among the two hypermultiplets TM-I and TM-II, only the latter can represent the (4,4) Liouville multiplet because of the $SU(2)$ structure of their leading components. It becomes obvious by noticing that the (4,4) supergravity constraints (17) are invariant under the (4,4) super-Weyl transformations having the form [14, 11]

$$\nabla_{\pm i} \to \frac{1}{2} P \, \nabla_{\pm i} \mp i L_i{}^j \, \nabla_{\pm j} \mp (\nabla_{\pm i} P) \mathcal{M} + (\nabla_{\pm j} P) \mathcal{Y}_i^j \ , \tag{21}$$

where the super-Weyl infinitesimal (4,4) superfield parameters $(P, L_i{}^j)$ form a TM-II multiplet.

The auxiliary fields of TM-II can be considered as the leading components of a TM-I which can be called the kinetic multiplet, like in 4d. Therefore, TM-I and TM-II are dual to each other, though they are not equivalent [12]. It implies, in particular, that there exists the supersymmetric invariant given by a product of TM-I and TM-II, without the introduction of a central charge [4]. In a curved superspace, this invariant takes the form

$$I = \int d^2x d^4\theta d^4\bar{\theta} E^{-1} (\Pi S + \Xi T) + \left[\int d^2x d^4\theta \mathcal{E}^{-1} \Lambda R + \text{h.c.} \right] \ , \tag{22}$$

where the (4,4) supervielbein determinant E^{-1}, the chiral super-density $\mathcal{E}^{-1}$, the real superfield prepotentials Π and Ξ, and the chiral superfield prepotential Λ of the TM-II have been introduced [12].

The rigidly (4,4) supersymmetric invariant describing the free TM-II action, which is quadratic in the fields, is known [12]. However, its locally (4,4) supersymmetric generalization does not exist. [4] The allowed matter couplings in the (4,4) conformal supergravity are much more restricted than in the rigid (4,4) case, and it is also known

[4] The same is true in 4d.

to be the case for the N=2 matter couplings in 4d, N=2 supergravity. [5] Fortunately, as far as the TM-II in 2d is concerned, this problem is only apparent, since there exists its *improved* 2d version, which can be coupled to the 2d, (4,4) conformal supergravity ! Namely, it is possible to form the TM-I out of the TM-II components in yet another *non-linear* way:

$$\begin{aligned} R_{\text{impr.}} &= L^{-1}\left(G - 2L_i{}^j A_j{}^i\right) - 2i\lambda_{\underset{i+}{\bullet}}\lambda^j{}_{\underset{-}{\bullet}}L_j{}^i L^{-3} \ , \\ S_{\text{impr.}} &= L^{-1}\left(M - 4PA\right) - i\left(\lambda_{\underset{i+}{\bullet}}\lambda^j{}_{-} - \lambda_{\underset{i-}{\bullet}}\lambda^j{}_{+}\right)L_j{}^i L^{-3} \ , \\ T_{\text{impr.}} &= L^{-1}\left(N - 4PA\right) + \left(\lambda_{\underset{i+}{\bullet}}\lambda^j{}_{-} + \lambda_{\underset{i-}{\bullet}}\lambda^j{}_{+}\right)L_j{}^i L^{-3} \ , \end{aligned} \tag{23}$$

where

$$L \equiv \sqrt{L_i{}^j (L_i{}^j)^*} \ . \tag{24}$$

Eq. (22) can now be used to define the invariant coupling of the improved TM-II to the (4,4) supergravity, in the form

$$I_{\text{impr.}} = \int d^2x d^4\theta d^4\bar{\theta} E^{-1}(\Pi S_{\text{impr.}} + \Xi T_{\text{impr.}}) + \left[\int d^2x d^4\theta \mathcal{E}^{-1}\Lambda R_{\text{impr.}} + \text{h.c.}\right] \ . \tag{25}$$

The existence of the improved TM-II in 2d is a direct consequence of the existence of the improved N=2 tensor multiplet in 4d [16], since they are related via dimensional reduction. Unlike the improved N=2 tensor multiplet in 4d, its 2d counterpart does not have any gauge degrees of freedom, which allows the (4,4) locally supersymmetric component action associated with eq. (25) to have manifest $SU(2)$ internal symmetry.

The action (25) is highly non-linear even in the absence of supergravity. The kinetic terms of the improved TM-II in eq. (25) are given by the non-linear sigma-model (NLSM) *with* torsion. A direct calculation yields

$$L_{NLSM} = -\frac{1}{2}g_{ab}\partial_\alpha X_a \partial^\alpha X_b - \frac{1}{2}i\varepsilon^{\alpha\beta}e_{ab}\partial_\alpha X_a \partial_\beta X_b \ , \tag{26}$$

where the symmetric metric $g_{ab}(X)$ and antisymmetric torsion potential $e_{ab}(X)$ are given by

$$g_{ab} = e^{-\phi}\delta_{ab} \ , \qquad e_{IJ} = -X_4\varepsilon_{IJK}X_K e^{-3\phi} \ , \quad e_{I4} = 0 \ , \tag{27}$$

[5] See, e.g., ref. [15] for a recent review.

when using the notation $a, b = I, 4$, and $I, \ldots = 1, 2, 3$, and

$$L_{ij} = \begin{pmatrix} X_1 + iX_2 & iX_3 \\ iX_3 & X_1 - iX_2 \end{pmatrix} , \quad P = X_4 \ , \quad L = \sqrt{X_1^2 + X_2^2 + X_3^2} \equiv e^{\phi} \ . \quad (28)$$

The NLSM of eq. (26) is quite remarkable. First, its four-dimensional target space has the explicit $SO(3) \simeq SU(2)$ internal symmetry. Second, the target space metric is conformally flat, and it does not depend on X_4. Third, despite of the apparent dependence of the torsion potential e_{ab} upon all of the coordinates, the geometrical torsion $H_{abc} = \frac{3}{2}\partial_{[a}e_{bc]}$ does *not* actually depend upon X_4, and, hence, the NLSM target space has an abelian isometry. Third and the most importantly, the torsion H_{abc} is just the *parallelizing* torsion for the curvature associated with the metric, i.e. the generalized Riemann curvature tensor (with torsion) identically zero in this case. Since the group manifolds are the only parallelizable manifolds in four dimensions, the symmetry considerations dictate that eq. (26) is just the 2d Wess-Zumino-Novikov-Witten (WZNW) model over the $SU(2) \otimes U(1)$ in the rather unusual parametrization of the target space !

We are now in a position to construct the manifestly locally (4,4) supersymmetric Liouville action. The quaternionic curvature of the (4,4) supergravity enters the action (25) linearly, being multiplied by the physical scalar components of the improved TM-II modulo (4,4) super-Weyl rescaling of the supergravity fields. Therefore, we only need to find a proper term which would generate the Liouville-type potential. As is well-known, there is, in fact, the additional natural resource to build a supersymmetric invariant, namely, the Fayet-Iliopoulos (FI) term. [6] In (4,4) superspace, the FI term takes the form

$$I_{\rm FI} = -2\mu \int d^2x d^4\theta d^4\bar{\theta} E^{-1} \Pi \ . \quad (29)$$

Given the action

$$I_4 = I_{\rm impr.} + I_{\rm FI} \ , \quad (30)$$

the auxiliary field P of the improved TM-II enters this action in the combination $e^{-\phi}P^2 - 2\mu P$. Eliminating the auxiliary field P via its algebraic equation of motion

[6] This term was also used in ref. [4] to construct the rigidly (4,4) supersymmetric Liouville action.

just yields the desired contribution $-\mu^2 e^{\phi}$. We can therefore conclude that the action (30) is the (4,4) locally supersymmetric Liouville action indeed.

4 *Conclusion.* It is well-known that the $SU(2) \otimes U(1)$ WZNW model has the classical (4,4) superconformal invariance, which is represented by the 'large' N=4 superconformal algebra in the left- or right-moving sector [17]. The (4,4) conformal supergravity respects only the 'small' $SU(2)$ part of this N=4 algebra. The fact that the kinetic terms of the improved TM-II action are given by the WZNW model allows one to apply the methods of conformal field theory [7] for an investigation of the quantum (4,4) Liouville theory. In ref. [6], this fact was not proved but used as the basic assumption.

For a detailed analysis of the (4,4) Liouville theory in a curved superspace of the (4,4) supergravity, one needs to develop further the superspace formalism, e.g. to solve the superspace constraints and to calculate the component projection formulae, which we postpone for another publication [11]. The related quantum aspects of the N=4 Liouville theory also deserve further studies [19].

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